

Analysis of Velocity Derivatives in Turbulence based on Generalized Statistics

N. ARIMITSU¹ and T. ARIMITSU²

¹ Graduate School of EIS, Yokohama Nat'l. University - Kanagawa 240-8501, Japan

² Institute of Physics, University of Tsukuba - Ibaraki 305-8571, Japan

PACS. 47.27.-i – First pacs description.

PACS. 47.53.+n – Second pacs description.

PACS. 47.52.+j – Third pacs description.

Abstract. –

A theoretical formula for the probability density function (PDF) of *velocity derivatives* in a fully developed turbulent flow is derived with the multifractal aspect based on the generalized measures of entropy, i.e., the extensive Rényi entropy or the non-extensive Tsallis entropy, and is used, successfully, to analyze the PDF's observed in the direct numerical simulation (DNS) conducted by Gotoh et al.. The minimum length scale r_d/η in the longitudinal (transverse) inertial range of the DNS is estimated to be $r_d^L/\eta = 1.716$ ($r_d^T/\eta = 2.180$) in the unit of the Kolmogorov scale η .

In the previous paper [1], we analyzed, precisely, the probability density functions (PDF's) of velocity fluctuations observed in the direct numerical simulation (DNS) of turbulence conducted by Gotoh et al. [2] at the Taylor microscale Reynolds number $R_\lambda = 381$. At this Reynolds number, the PDF's had been measured with high accuracy up to the order of $10^{-9} \sim 10^{-10}$. The correctness of the analytical formula for the scaling exponents of velocity structure function [3–6] enabled us to extract the value of the intermittency exponent μ by fitting it with the ten observed data in the DNS by the method of least square [1]. With the intermittency exponent, the parameters in the analytical formula of the PDF of velocity fluctuations [6–8] are determined, self-consistently. The formula was used to fit each of the observed PDF's of velocity fluctuations for ten different separations r/η by means of the method of least square [1], and to extract, successfully, the dependence of the number n of steps in the energy cascade on the separation r/η :

$$n = -1.050 \times \log_2 r/\eta + 16.74 \quad (\text{for } \ell_c^L \leq r), \quad (1)$$

$$n = -2.540 \times \log_2 r/\eta + 25.08 \quad (\text{for } r < \ell_c^L) \quad (2)$$

with the crossover length $\ell_c^L/\eta = 48.26$ for *longitudinal* fluctuations, and

$$n = -0.9896 \times \log_2 r/\eta + 13.95 \quad (\text{for } \ell_c^T \leq r), \quad (3)$$

$$n = -2.820 \times \log_2 r/\eta + 23.87 \quad (\text{for } r < \ell_c^T) \quad (4)$$

with the crossover length $\ell_c^T/\eta = 42.57$ for *transverse* fluctuations ⁽¹⁾. These straight lines on semi-logarithmic sheet, explicitly, told us that there exist two scaling regions [1], i.e., the *upper* scaling region with larger separations which may correspond to the scaling range observed by Gotoh et al. [2], and the *lower* scaling region with smaller separations which is another scaling region extracted first by the systematic analyses in [1]. These scaling regions are divided by the crossover lengths ℓ_c^L/η and ℓ_c^T/η approximately of the order of the Taylor microscale $\lambda/\eta = 38.33$ reported in [2] at $R_\lambda = 381$.

We can view that the turbulent flow, satisfying the Navier-Stokes equation

$$\partial \vec{u} / \partial t + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\vec{\nabla} (p/\rho) + \nu \nabla^2 \vec{u} \quad (5)$$

of an incompressible fluid, consists of a cascade of eddies with different diameter of the order of $\ell_n = \delta_n \ell_0$ where $\delta_n = 2^{-n}$ ($n = 0, 1, 2, \dots$). The quantities ρ and p represent, respectively, the mass density and the pressure. At each step of the cascade, say at the n th step, mother-eddies of size ℓ_{n-1} produce smaller daughter-eddies having a half of mother's diameter with the energy-transfer rate ϵ_n that represents the rate of transfer of energy per unit mass from eddies of size ℓ_{n-1} to those of size ℓ_n (the energy cascade model). Following formally the energy cascade model, we are measuring space scale by ℓ_n . However our analysis in the following is not restricted within the energy cascade model, i.e., the number of step n can be real number.

In this paper, we will derive the formula for the PDF of *velocity derivatives* in fully developed turbulence by the statistics based on the generalized entropy, i.e., Rényi's [10] or Tsallis' [11, 12], and will analyze the velocity derivative PDF's obtained in the DNS [2] at $R_\lambda = 381$ having far better accuracy than that in any previous experiments, real or numerical. The main interest, here, is the *longitudinal* velocity derivative $\partial u(\vec{r}) / \partial r_1$ and the *transverse* velocity derivative $\partial u(\vec{r}) / \partial r_2$ (or $\partial u(\vec{r}) / \partial r_3$), where $\vec{r} = (r_1, r_2, r_3)$, and u is the r_1 -component of the fluid velocity field \vec{u} of the turbulent flow produced by a grid with size ℓ_0 putting in a laminar flow parallel to the r_1 direction. Introducing the velocity difference δu_n of the component u at two points separated by the distance ℓ_n , the velocity derivatives may be estimated by $\lim_{\ell_n \rightarrow 0} \delta u_n / \ell_n$ ($= \lim_{n \rightarrow \infty} \delta u_n / \ell_n$). The Reynolds number Re of the system is given by $Re = \delta u_0 \ell_0 / \nu = (\ell_0 / \eta)^{4/3}$. For high Reynolds number $Re \gg 1$, or for the situation where effects of the kinematic viscosity ν can be neglected compared with those of the turbulent viscosity, the Navier-Stokes equation (5) is invariant under the scale transformation [13, 14]: $\vec{r} \rightarrow \lambda \vec{r}$, $\vec{u} \rightarrow \lambda^{\alpha/3} \vec{u}$, $t \rightarrow \lambda^{1-\alpha/3} t$ and $(p/\rho) \rightarrow \lambda^{2\alpha/3} (p/\rho)$. The exponent α is an arbitrary real quantity which specifies the degree of singularity in the velocity derivative [15] for $\alpha < 3$, i.e., $\lim_{\ell_n \rightarrow 0} \delta u_n / \ell_n \sim \lim_{\ell_n \rightarrow 0} \ell_n^{\alpha/3-1}$ which can be seen with the relation $\delta u_n / \delta u_0 = (\ell_n / \ell_0)^{\alpha/3}$.

The energy $E_n = (\delta u_n)^2 / 2$ per unit mass contained in an eddy of size ℓ_n is estimated as $E_n \sim (\ell_n \omega_n)^2$ where $\omega_n \sim \delta u_n / \ell_n$ represents the angular momentum of the eddy. We see that $\lim_{\ell_n \rightarrow 0} \omega_n$ has the same singularity as the velocity derivative does. Within the region satisfying the scale invariance, we have $E_n = E_0 \delta_n^{2\alpha/3}$. The energy spectrum $E(k)$, defined through $E_n = \int_{k_n}^{k_{n+1}} dk E(k)$ with $k_n = \ell_n^{-1}$, has the wavenumber dependence $E(k) \propto k^{-1-2\alpha/3}$. By the way, the energy transfer rate, estimated by $\epsilon_n \sim E_n \omega_n$, satisfies $\epsilon_n = \epsilon_0 \delta_n^{\alpha-1}$ in the region of the scale invariance. Kolmogorov's assumption in K41 [9] that there is no fluctuation in ϵ_n leads us to $\alpha = 1$. With this value of α , $E(k)$ represents the Kolmogorov energy spectrum, i.e., $E(k) \propto k^{-3/5}$. In order to explain the intermittency in turbulence, we shall introduce a fluctuation in α .

⁽¹⁾ Here, η is the Kolmogorov scale [9] defined by $\eta = (\nu^3 / \epsilon)^{1/4}$ with ν being the kinematic viscosity, and ϵ ($= \epsilon_0$) the energy input rate to the largest eddies with size ℓ_0 , and has the value $\eta = 2.58 \times 10^{-3}$ in the DNS [2] at $R_\lambda = 381$.

The present analysis rests on the assumption that the distribution of the exponent α is multifractal, and that the probability $P^{(n)}(\alpha)d\alpha$ to find, at a point in physical space, an eddy of size ℓ_n having a value of the degree of singularity in the range $\alpha \sim \alpha + d\alpha$ is given by $P^{(n)}(\alpha) = [P^{(1)}(\alpha)]^n$ with [3–6]

$$P^{(1)}(\alpha) \propto [1 - (\alpha - \alpha_0)^2 / (\Delta\alpha)^2]^{1/(1-q)}, \quad (\Delta\alpha)^2 = 2X / [(1-q) \ln 2]. \quad (6)$$

Here, it is assumed that each step in the cascade is statistically independent. The distribution function (6) is derived by taking an extremum of the generalized entropy (2) with the two constraints, i.e., the normalization of distribution function: $\int d\alpha P^{(1)}(\alpha) = \text{const.}$ and the q -variance being kept constant as a known quantity: $\sigma_q^2 = (\int d\alpha P^{(1)}(\alpha)^q (\alpha - \alpha_0)^2) / \int d\alpha P^{(1)}(\alpha)^q$.

The dependence of the parameters α_0 , X and q on the intermittency exponent μ is determined, self-consistently, with the help of the three independent equations, i.e., the energy conservation: $\langle \epsilon_n \rangle = \epsilon$, the definition of the intermittency exponent μ : $\langle \epsilon_n^2 \rangle = \epsilon^2 \delta_n^{-\mu}$, and the scaling relation (3): $1/(1-q) = 1/\alpha_- - 1/\alpha_+$ with α_{\pm} satisfying $f(\alpha_{\pm}) = 0$ where the multifractal spectrum [3–6]

$$f(\alpha) = 1 + (1-q)^{-1} \log_2 \left[1 - (\alpha - \alpha_0)^2 / (\Delta\alpha)^2 \right] \quad (7)$$

is derived by the relation $P^{(n)}(\alpha) \propto \delta_n^{1-f(\alpha)}$ [6,13] that reveals how densely each singularity, labeled by α , fills physical space. The average $\langle \dots \rangle$ is taken with $P^{(n)}(\alpha)$. Note that the relation between α and ϵ_n is given by $\epsilon_n/\epsilon = (\ell_n/\ell_0)^{\alpha-1}$. For the region where the value of μ is usually observed, i.e., $0.13 \leq \mu \leq 0.40$, the three self-consistent equations are solved to give the approximate equations [1]: $\alpha_0 = 0.9989 + 0.5814\mu$, $X = -2.848 \times 10^{-3} + 1.198\mu$ and $q = -1.507 + 20.58\mu - 97.11\mu^2 + 260.4\mu^3 - 365.4\mu^4 + 208.3\mu^5$.

Let us suppose that ℓ_d is the typical length giving the minimum scale within the energy cascade model. With this shortest length ℓ_d , the velocity derivatives may be given by $|s| = \delta u_d / \ell_d$ with $\delta u_d = \delta u_{n=n_d}$ where n_d is introduced through $\ell_d / \ell_0 = \delta_d = 2^{-n_d}$. Since there are two mechanisms in turbulent flow to rule its dissipative evolution, i.e., the one controlled by the kinematic viscosity that takes care thermal fluctuations, and the other by the turbulent viscosity that is responsible for intermittent fluctuations related to the singularities in velocity derivative, it may be reasonable to assume that the probability $\Pi(x)dx$ to find the scaled velocity derivative $x = \delta_d t_0 s$ in the range $x \sim x + dx$ can be divided into two parts:

$$\Pi(x)dx = \Pi_S(|x|)dx + \Delta\Pi(x)dx. \quad (8)$$

Here, the singular part PDF $\Pi_S(|x|)$ represents the contribution from multifractal distribution of the singularities, and the correction part $\Delta\Pi(x)$ from the viscous term neglected in the scale transformation. The former is derived through $\Pi_S(|x|)dx = P^{(n)}(\alpha)d\alpha$ with the transformation of the variables: $|x| = \delta_d^{\alpha/3}$. The m th moments of the velocity derivatives, defined by $\langle |x|^m \rangle = \int_{-\infty}^{\infty} dx |x|^m \Pi(x)$, are given by

$$\langle |x|^m \rangle = 2\gamma_m + (1 - 2\gamma_0) a_m \delta_d^{\zeta_m} \quad (9)$$

(2) The Rényi entropy $S_q^R[P^{(1)}(\alpha)] = (1-q)^{-1} \ln \int d\alpha P^{(1)}(\alpha)^q$ [10] has the extensive character as the usual thermodynamical entropy does, whereas the Tsallis entropy $S_q^T[P^{(1)}(\alpha)] = (1-q)^{-1} (\int d\alpha P^{(1)}(\alpha)^q - 1)$ [11, 12, 16] is non-extensive. In spite of different characteristics of these entropies the distribution functions giving the extremum of each entropy have the common structure (6).

(3) The scaling relation is a generalization of the one derived first in [17, 18] to the case where the multifractal spectrum has negative values.

with $a_{3\bar{q}} = \{2/[C_{\bar{q}}^{1/2}(1+C_{\bar{q}}^{1/2})]\}^{1/2}$, $C_{\bar{q}} = 1 + 2\bar{q}^2(1-q)X \ln 2$ and $2\gamma_m = \int_{-\infty}^{\infty} dx |x|^m \Pi_N(x)$. We used the normalization: $\langle\langle 1 \rangle\rangle = 1$. The quantity

$$\zeta_m = \alpha_0 m/3 - 2Xm^2 / \left[9 \left(1 + C_{m/3}^{1/2} \right) \right] - \left[1 - \log_2 \left(1 + C_{m/3}^{1/2} \right) \right] / (1-q) \quad (10)$$

is the so-called scaling exponent of the velocity structure function, whose expression was derived first by the present authors [3–6]. Note that the formula is independent of the length ℓ_d , and, therefore, independent of ℓ_n .

	longitudinal	transverse
μ	0.240	0.327
q	0.391	0.543
α_0	1.138	1.189
X	0.285	0.388

TABLE I – *Values of the intermittency exponent μ and the parameters q , α_0 and X for longitudinal and transverse velocity fluctuations [1], determined by the formula (10) being consistent with the observed data in DNS conducted by Gotoh et al..*

With the help of the analytical formula (10), we determined in [1] the values of the intermittency exponent μ and the parameters q , α_0 and X by fitting the ten DNS data of the scaling exponents ζ_m ($m = 1, 2, \dots, 10$) at $R_\lambda = 381$ [2] with the method of least squares. The determined values are listed in table I both for the longitudinal and transverse velocity fluctuations. Note that the relation $\mu = 2 - \zeta_6$ is satisfied within the experimental error bars. We have $\alpha_+ - \alpha_0 = \alpha_0 - \alpha_- = 0.6818$ (0.8167), $\Delta\alpha = 1.160$ (1.566) for the longitudinal (transverse) fluctuations.

Since we are interested in the large deviation stemmed from the singular part $\Pi_S(|x|)$ that may contribute to the symmetric part of the PDF, we will symmetrize right and left of the experimental PDF, and will compare it with our theoretical PDF in the following.

Let us introduce the PDF $\hat{\Pi}(|\xi|)$ of the velocity derivatives by $\hat{\Pi}(|\xi|)d\xi = \Pi(|x|)dx$ with the new variable

$$\xi = s/\langle\langle s^2 \rangle\rangle^{1/2} = x/\langle\langle x^2 \rangle\rangle^{1/2}, \quad |\xi| = \bar{\xi} \delta_d^{\alpha/3 - \zeta_2/2}, \quad (11)$$

scaled by the variance of velocity derivatives. It may be appropriate to devide the PDF into two parts:

$$\hat{\Pi}(\xi) = \hat{\Pi}_{*<}(\xi) \quad \text{for } \xi^* \leq |\xi| \leq \bar{\xi} \delta_d^{\alpha_{\min}/3 - \zeta_2/2} \quad (12)$$

$$\hat{\Pi}(\xi) = \hat{\Pi}_{<*}(\xi) \quad \text{for } |\xi| \leq \xi^* \quad (13)$$

with the point ξ^* defined by $\xi^* = \bar{\xi} \delta_d^{\alpha^*/3 - \zeta_2/2}$ where α^* is the solution of $\zeta_2/2 - \alpha/3 + 1 - f(\alpha) = 0$ that provides us with the least n_d -dependence of $\hat{\Pi}(\xi^*)$. Here, $\bar{\xi} = [2\gamma_2 \delta_d^{-\zeta_2} + (1 - 2\gamma_0) a_2]^{-1/2}$. As the value of ξ^* turns out to be of order 1 (see below), we are deviding the region by the order of the variance of velocity derivative. $\hat{\Pi}_{<*}(\xi)$ and $\hat{\Pi}_{*<}(\xi)$ are connected at ξ^* under the condition that they should have the same value and the same slope there. In our analysis, since we are assuming that the large deviations of the velocity derivative come from the multifractal distribution of these singularities in real space [15], it is consistent to put $\hat{\Pi}_{*<}(\xi) = \hat{\Pi}_S(|\xi|)$. This leads

$$\hat{\Pi}_{*<}(\xi) = \bar{\Pi}_S(\bar{\xi}/|\xi|) \left\{ 1 - \left[3 \ln |\xi/\xi_0| / (\Delta\alpha |\ln \delta_d|) \right]^2 \right\}^{n_d/(1-q)} \quad (14)$$

with $|\xi_0| = \bar{\xi} \delta_d^{\alpha_0/3 - \zeta_2/2}$. For smaller velocity derivatives $|\xi| \leq \xi^*$, the contribution to the PDF may come, mainly, from the term ignored in the scale transformation producing, i.e., the term responsible for thermal fluctuations related to the kinematic viscosity, and also from measurement errors, we take for the PDF $\hat{H}_{<*}(\xi)$ a Gaussian function. The connection at ξ^* with (14) gives us

$$\hat{H}_{<*}(\xi) = \bar{H}_S e^{-(1+3f'(\alpha^*))[(\xi/\xi^*)^2 - 1]/2} \quad (15)$$

with $\bar{H}_S = 3(1 - 2\gamma_0)/(2\bar{\xi}\sqrt{2\pi X |\ln \delta_d|})$. It is remarkable that the PDF $\hat{H}(|\xi|)$ of the *velocity derivative*, given by (12) and (13) with (14) and (15), turns out to have the same structure as the PDF $\hat{H}^{(n)}(|\xi_n|)$ of the *velocity fluctuations* [1, 6–8] with the separation $\ell_n = \ell_d$.

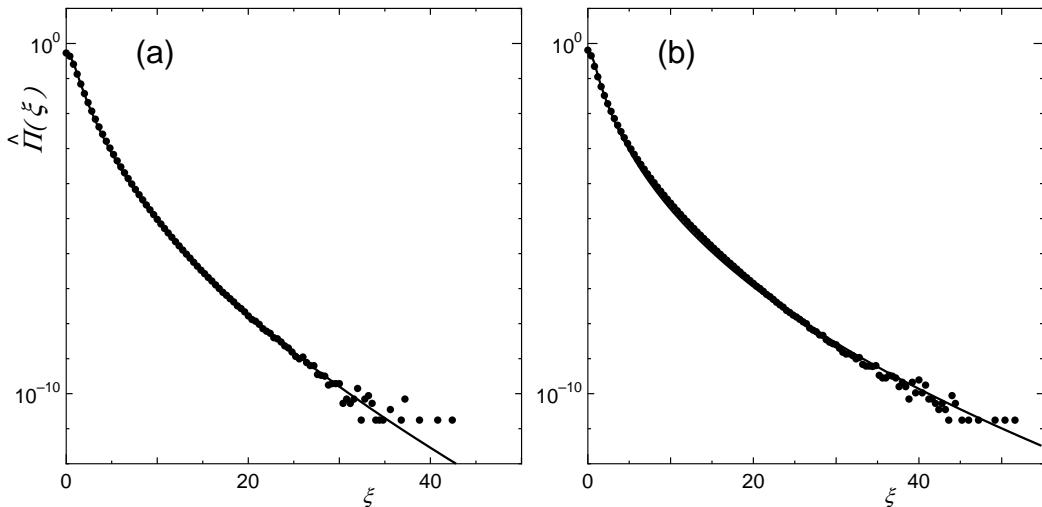


Fig. 1 – PDF's of the velocity derivatives for (a) longitudinal and for (b) transverse. Closed circles are the symmetrized points of the PDF observed in DNS conducted by Gotoh et al. at $R_\lambda = 381$ [2]. Solid line represents the PDF given by the analytic formula (12) and (13) with the parameters given in Table I, and with (a) $n_d = n_d^L = 23.1$ and (b) $n_d = n_d^T = 20.7$.

The comparison between the PDF's of the velocity derivatives at $R_\lambda = 381$ measured in the DNS [2] and those obtained by the present analysis is given in fig. 1 (a) for longitudinal derivative and in fig. 1 (b) for transverse derivative. In order to extract the symmetrical part of the PDF, we took mean average between the DNS data on the left hand side and that on the right hand side. The symmetrized data are described by closed circles. The solid lines are the curves of $\hat{H}(\xi)$ given by (14) and (15) with the values of parameters in table I. Note that $\xi^* = 0.982$ for the longitudinal velocity derivative, whereas $\xi^* = 0.900$ for the transverse derivative. The number $n_d^L = 23.1$ ($n_d^T = 20.7$) of steps in the cascade for longitudinal (transverse) velocity fluctuations is derived by the method of least squares with respect to the logarithm of PDF's for the best fit of our theoretical formulae, consisting of (14) and (15), to the observed values of the PDF by discarding those points which have observed values less than 10^{-10} since they scatter largely in the logarithmic scale. We see an excellent agreement between the measured PDF for the velocity derivatives and the analytical formula of PDF derived by the present self-consistent theory.

Substituting the obtained values $n_d^L = 23.1$ ($n_d^T = 20.7$) of the number of steps in the energy cascade for longitudinal (transverse) velocity fluctuations into (2) (into (4)), we obtain the shortest length r_d of separation in the inertial range with the value $r_d^L/\eta = 1.716$ ($r_d^T/\eta =$

2.180). Then, we can conclude that the range of the *lower* scaling region in the inertial range of the longitudinal (transverse) velocity fluctuations is given by $r_d^L/\eta \leq r/\eta \leq \ell_c^L/\eta$ ($r_d^T/\eta \leq r/\eta \leq \ell_c^T/\eta$). Adding to this the *upper* scaling region, the total inertial range for longitudinal (transverse) fluctuations within the DNS [2] at $R_\lambda = 381$ turns out to be $1.716 \leq r/\eta \leq 1220$ ($2.180 \leq r/\eta \leq 1220$), where the value 1220 is the largest separation taken by Gotoh et al. [2] for the measurement of the PDF's of velocity fluctuations. Note that the largest scale ℓ_0 for the longitudinal (transverse) fluctuations can be estimated by putting $n = 0$ into (1) (into (3)), i.e., $\ell_0^L/\eta = 6.299 \times 10^4$ ($\ell_0^T/\eta = 1.752 \times 10^4$) leading to $Re^L = 2.506 \times 10^6$ ($Re^T = 4.550 \times 10^5$). We see that the shortest length scale $\ell_d/\eta = 2^{-n_d} \ell_0/\eta$ for longitudinal (transverse) velocity fluctuations within the energy cascade model becomes $\ell_d^L/\eta = 7.006 \times 10^{-3}$ ($\ell_d^T/\eta = 1.029 \times 10^{-2}$) which is different from the length r_d^L/η (r_d^T/η) giving the estimate of the lowest end of inertial range in the DNS [1] at $R_\lambda = 381$. We presume, here, that r/η in the formulae (1), (2) and (3), (4) provides us with a *real* distance in the support of the velocity fields $\vec{u}(\vec{r})$.

Summarizing, we derived in this paper the formula for PDF of velocity derivatives, (12) and (13) with (14) and (15), and showed that it explains the observed PDF's in the DNS [2], precisely up to the order of 10^{-10} , with the parameters in Table I. The latter analysis provides us with the number n_d of steps in the energy cascade. The shortest length scale r_d^L (r_d^T), which serves the lowest end of the inertial range, is derived by making use of the obtained value of n_d^L (n_d^T) by assuming that the formula (2) (the formula (4)) between n and r/η , derived in the analyses of the PDF of longitudinal (transverse) velocity fluctuations [1], is applicable even for this shorter scale of length.

Let us close this paper by mentioning something about another trial for deriving the PDF of velocity derivatives. We introduced, as a preliminary test, the same philosophy for the definition of velocity derivative proposed by Benzi et al. [19] into the present analysis based on the generalized statistics, and saw that the PDF of velocity derivatives thus obtained (see (43) in [6]) cannot explain the PDF's provided in the DNS [2] so precise as the present PDF given by (12) and (13). A further investigation on the relation between the approach in [19] and the present one may be one of the interesting future problems.

The authors are grateful to Prof. T. Gotoh for enlightening discussion and his kindness to show his data prior to publication. The authors would like to thank Prof. C. Tsallis for his fruitful comments with encouragement.

REFERENCES

- [1] T. Arimitsu and N. Arimitsu, *J. Phys.: Condens. Matter*, **14**, 2237 (2002).
- [2] T. Gotoh, D. Fukayama and T. Nakano, *Phys. Fluids*, in press (2002).
- [3] T. Arimitsu and N. Arimitsu, *J. Phys. A: Math. Gen.* **33**, L235 (2000) [CORRIGENDUM: **34**, 673 (2001)].
- [4] T. Arimitsu and N. Arimitsu, *Chaos, Solitons and Fractals* **13**, 479 (2002).
- [5] T. Arimitsu and N. Arimitsu, *Prog. Theor. Phys.* **105**, 355 (2001).
- [6] T. Arimitsu and N. Arimitsu, *Physica A* **295**, 177 (2001).
- [7] N. Arimitsu and T. Arimitsu, *J. Korean Phys. Soc.* **40**, 1032 (2002).
- [8] T. Arimitsu and N. Arimitsu, *Physica A* **305**, 218 (2002).
- [9] A.N. Kolmogorov, *C.R. Acad. Sci. USSR* **30**, 301; 538 (1941).
- [10] A. Rényi, *Proc. 4th Berkeley Symp. Maths. Stat. Prob.* **1**, 547 (1961).
- [11] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).
- [12] C. Tsallis, *Braz. J. Phys.* **29**, 1 (1999); On the related recent progresses see at <http://tsallis.cat.cbpf.br/biblio.htm>.
- [13] C. Meneveau and K.R. Sreenivasan, *Nucl. Phys. B (Proc. Suppl.)* **2**, 49 (1987).

- [14] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, ed. by M. Ghil, R. Benzi and G. Parisi (North-Holland, New York, 1985) 84.
- [15] R. Benzi, G. Paladin, G. Parisi and A. Vulpiani, *J. Phys. A: Math. Gen.* **17**, 3521 (1984).
- [16] J.H. Havrda and F. Charvat, *Kybernetika* **3**, 30 (1967).
- [17] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, *Phys. Rev. E* **56**, 245 (1997).
- [18] M.L. Lyra and C. Tsallis, *Phys. Rev. Lett.* **80**, 53 (1998).
- [19] R. Benzi, L. Biferale, G. Paladin, A. Vulpiani and M. Vergassola, *Phys. Rev. Lett.* **67**, 2299 (1991).